10.5.3 Limiting magnitudes in Berek's model

We start with equation (9.5) of section 9.1.2, and note that, because of the tiny apparent angle of the target, its first term clearly dominates over the second term, so that the latter may be omitted. After taking the square we obtain the threshold-contrast

$$C = \frac{\phi(L_a)}{\sigma^2 L_a} , \tag{10.22}$$

in which we set the adaptive luminance equal to the background luminance L_{sky} of the night sky. The contrast between star and sky is then $C = (L_{star} - L_{sky})/L_{sky}$, and the luminance of a barely detectable star amounts to

$$L_{star} = \frac{\phi(L_{sky})}{\sigma^2} + L_{sky} . \tag{10.23}$$

At this point, a few peculiarities that apply to the detection of stars have to be considered: The apparent angle of any star disk is far below the resolution limit of both the unaided eye or the optical instrument. As a result, what is seen is never the star disk itself, but its diffraction disk, and when assuming a reasonably good imaging of the optical instrument, then the diameter of the star's image on the retina is invariant of the magnification. It always corresponds to a circular object at the resolution limit of the bare eye, roughly 1arcmin. As a result, the target angle σ in equation (10.23) will be set to unity in the remaining parts of this section.

The luminance of a star seen through the eyepiece is proportional to the luminous flux that passes through the instrument, thus $\sim D^2$, since the area of its image is invariant. As long as the light enters the eye without loss, because the exit pupil diameter d remains smaller than the eye pupil diameter d_e, the instrumental threshold luminance should therefore scale as $\tilde{L}_{star} = L_{star}/(D/d_e)^2$, because the amount of light on the retina has increased by the factor $(D/d_e)^2$. If, on the other hand, the exit pupil diameter exceeds the eye pupil diameter, then some of the light collected by the instrument is wasted, and we once again use the effective magnification m', being identical to $m' = m$ if $d > d_e$, or else $m' = D/d_a$. With these preparations we are arriving at a preliminary solution,

$$\tilde{L}_{star} \stackrel{?}{=} \frac{L_{star}}{(m')^2} , \tag{10.24}$$

hastingly adding that one contribution is still missing: Contrary to the star image, the background luminance of the night sky is in fact a function of the magnification. Once the exit pupil is smaller than the eye pupil, the background is turning increasingly dark while the exit pupil diameter is shrinking. This leads to

$$\tilde{L}_{sky} = L_{sky} \left(\frac{m'}{m}\right)^2 = \left(\frac{D}{d_a m}\right)^2 . \tag{10.25}$$

The following facts have to be considered here: Compared to the unaided eye, the instrument collects an amount of background light that is larger by a factor $(D/d_a)^2$. That light is then spread over an area that is larger by the factor m^2, thus defining the instrumental luminance level of the background. If the astronomer is sufficiently patient, then her adaptive luminance is approaching the further darkened sky of luminance $\tilde{L}_{sky}$, allowing her to discern stars of somewhat lower intensities. The correct threshold luminance is then turning into

$$\tilde{L}_{star} = \frac{\phi(\tilde{L}_{sky}) + \tilde{L}_{sky}}{m'^2} , \tag{10.26}$$

and we are defining Berek's astro index as the inverse of that threshold luminance, yielding

$$I_B = \tilde{L}_{star}^{-1} = \frac{m'^2}{\phi(\tilde{L}_{sky}) + \tilde{L}_{sky}} . \quad \text{(Berek index)} \tag{10.27}$$

We shall now compare the indices of two binoculars with the objective diameters $D_2 = 2D_1$ and